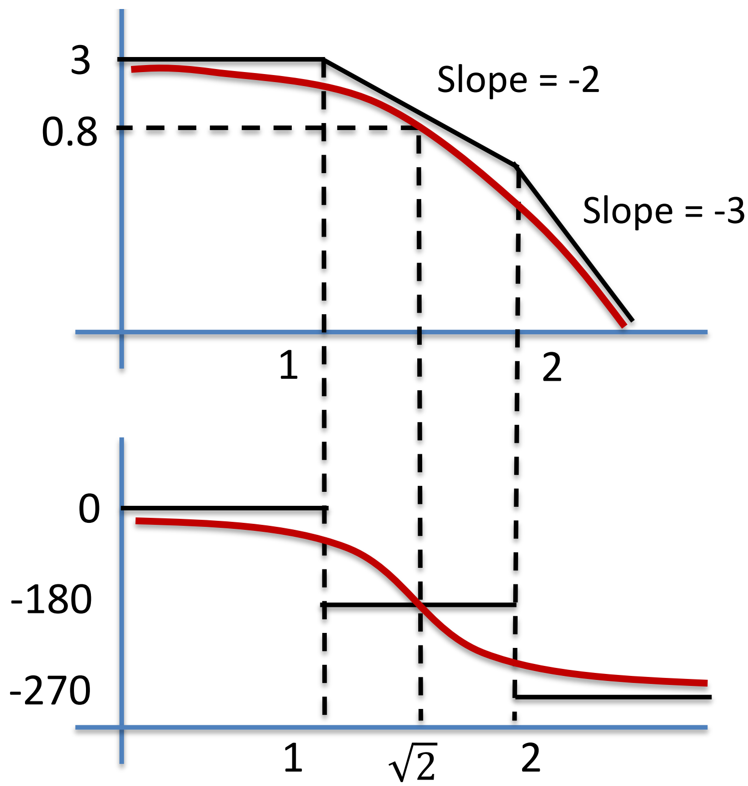
ME-221
SOLUTIONS FOR PROBLEM SET 12

Problem 1

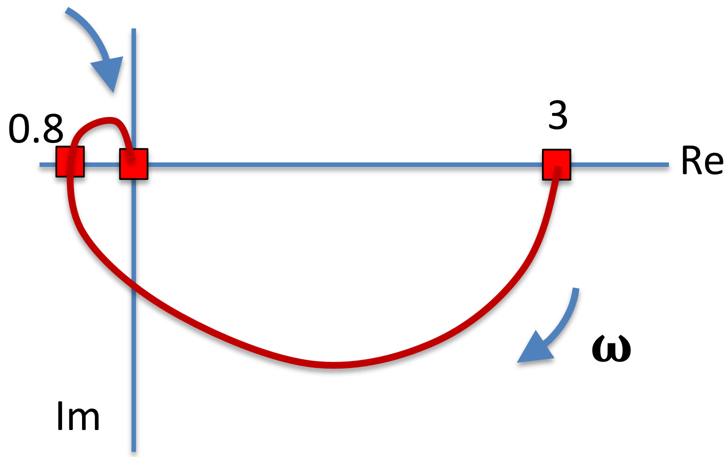
The transfer function is given by

$$G(s) = \frac{6}{(s+1)^2(s+2)}$$

We have a repeated pole at -1 and a pole at -2. We first sketch the Bode plot of this system.



The next step is the generation of the Nyquist plot. At very low frequency, the phase is $0°$ and the magnitude is 3. As the frequency is increased, the magnitude will monotonically decrease while the phase angle will move the position in the complex plane; first in the fourth quadrant, then in the third quadrant, and then finally in the second quadrant until the magnitude becomes zero at $-270°$. We can find a few points in the complex plane to better sketch the Nyquist plot. For instance, at $\omega = \sqrt{2}$, the phase angle is $-180°$. The magnitude at this frequency is $6/(3\sqrt{6}) = 0.816$. Another point would be $\omega = 1$, at which magnitude is $6/(2\sqrt{5}) = 1.4$ and the phase angle is $-90° - 26° = -116°$. The plot is as follows:



Problem 2

a) The slope of the first part of the magnitude curve is -1 while the second part has a slope of -3. Phase angle starts from $-90°$ and asymptotically converges to $-270°$. Thus, the transfer function must have the following form:

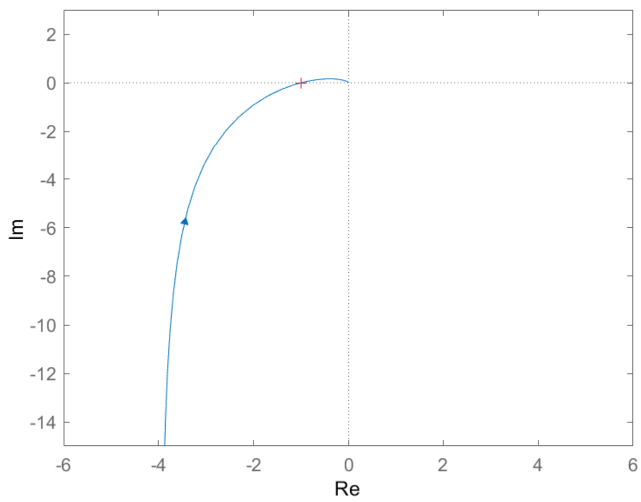
$$G(s) = \frac{K}{s(\tau s + 1)^2}$$

The second order term cannot have distinct poles as there is only one corner frequency. Due to lack of resonance, we consider the system to be critically damped with a repeated pole. From the magnitude plot, we can find the values of K and τ. The slope of the asymptotic curves change at $\omega = 1$ and $\tau = 1/\omega = 1$. At the corner frequency, the magnitude of the sinusoidal transfer function is K, which is 2. The transfer function is then given by:

$$G(s) = \frac{2}{s(s + 1)^2}$$

The order of the system is 3. There is a repeated pole at $p = -1$ and another pole at the origin.

b) The Nyquist plot is:



c) The filter has the following form:

$$F(s) = \frac{K_f(s+1)}{\tau_f s + 1}$$

The transfer function of the filtered system is then:

$$G_f(s) = \frac{2K_f}{s(\tau_f s + 1)(s+1)}$$

The magnitude of $G_f(j\omega)$ is given by:

$$|G_f(j\omega)| = \frac{2K_f}{\omega\sqrt{1 + (\tau_f \omega)^2}\sqrt{1+\omega^2}} \tag{1}$$

At $\omega = 1$, the magnitude is expected to be 1. So,

$$|G_f(j\omega)| = \frac{2K_f}{\sqrt{1+\tau_f^2}\sqrt{2}} = 1 \tag{2}$$

The phase angle is given by:

$$arg(G_f(j\omega)) = -90^\circ - arctan(\tau_f \omega) - arctan(\omega) \tag{3}$$

At $\omega = 1$, the phase is expected to be $-145°$. So,

$$arg(G_f(j\omega)) = -90^\circ - arctan(\tau_f) - arctan(1) = -145^\circ \tag{4}$$

From (4), we can find the value of the time constant as $\tau_f = 0.176$. If we use the value of τ_f in equation (2), we can find the gain $K_f = 0.718$. The transfer function of the filter is found as:

$$F(s) = \frac{0.718(s+1)}{0.176s+1}$$

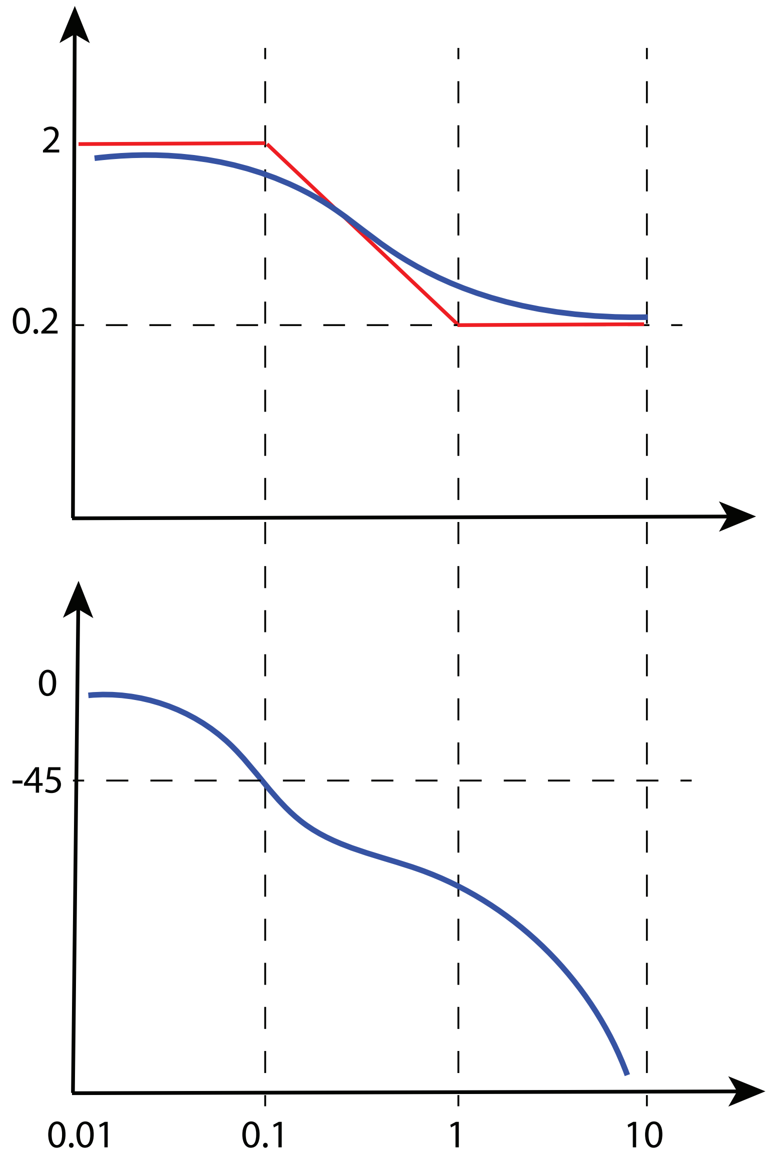
Problem 3

The transfer function is given by:

$$G(s) = \frac{2(s+1)}{10s+1}e^{-2s}$$

We have a pole at $p = -0.1$ and a zero at $z = -1$. At the first corner frequency ($\omega = 0.1$) the slope of the magnitude curve will go from 0 to -1 and at the second corner frequency ($\omega = 1$) the slope will go back to 0. The exponential term has no effect on the magnitude plot as its magnitude is 1. The phase plot will start from $0°$. If we did not have the exponential term, due to the pole, we would expect a decrease to $-90°$ where the angle is $-45°$ at $\omega = 0.1$. The zero would eventually move the angle back to $0°$ where the phase angle is expected to be $-45°$ at $\omega = 1$. However, the phase angle of the exponential term is -2ω. Thus, the phase angle will monotonically decrease for increasing frequency.

The Bode plot is shown below.



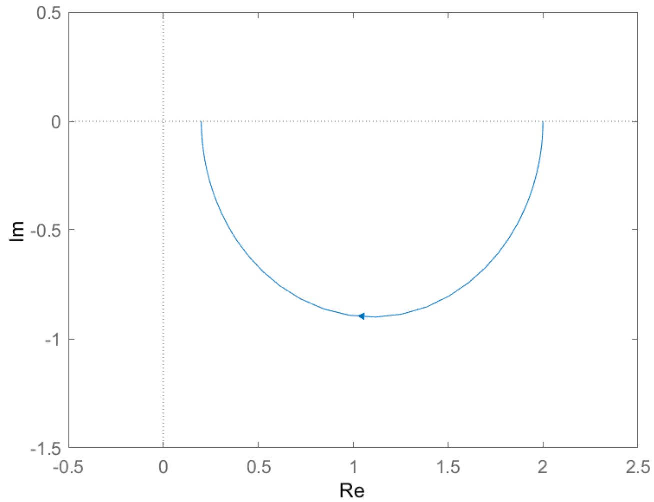
b) The input is given as $u(t) = 10sin(t)$. We need to find the magnitude and phase of the sinusoidal transfer function at $\omega = 1$.

$$|G(j\omega)| = \frac{2\sqrt{1+\omega^2}}{\sqrt{1+100\omega^2}} \tag{5}$$

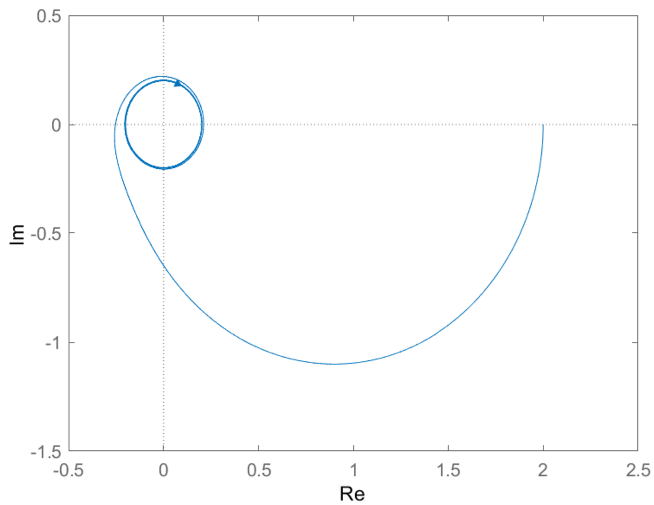
$$arg(G(j\omega)) = \phi(\omega) = arctan(\omega) - arctan(10\omega) - 2\omega \tag{6}$$

The magnitude is 0.28 and the phase angle is -2.68 rad at $\omega = 1$. The amplitude of the output is the multiplication of the magnitude of the sinusoidal transfer function and the amplitude of the input, which is given as 10. The output is $y(t) = 2.8sin(t - 2.68)$.

c) Nyquist Plot without the delay is:



With the 2s delay, it becomes:



d) The transfer function of the filtered system is:

$$G_f(s) = \frac{2(\alpha s + 1)}{10s + 1} e^{-2s}$$

According to the specifications, the phase angle at $\omega = 1$ must be $-120°$.

$$arg(G(j\omega)) = \phi(\omega = 1) = arctan(\alpha) - arctan(10) - 2(\frac{180°}{\pi}) = -120° \quad (7)$$

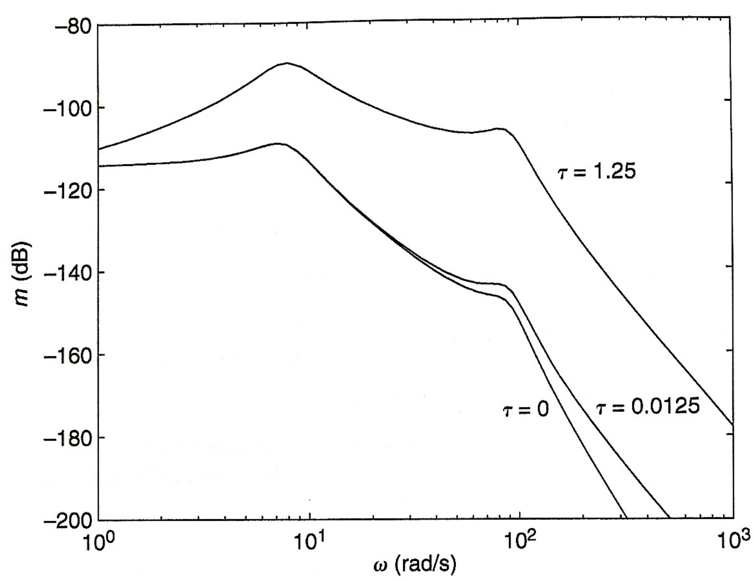
We can then calculate α, which is 5.1. The transfer function of the filter is:

$$F(s) = \frac{5.1s + 1}{s + 1}$$

Problem 4

The system is 4th order and it has two pairs of complex conjugate poles as well as a zero. The first second order term has $\omega_0 = 8$ rad/s and $\zeta = 0.3$. The second second order term has $\omega_0 = 90$ rad/sec and $\zeta = 0.2$. Both magnitude plots are expected to display resonance as $\zeta < 0.707$. The resonance frequencies corresponding to the two root pairs are $\omega_{r1} = 7.24$ rad/s and $\omega_{r2} = 86.3$ rad/s. However, we cannot use the standard formula to calculate the magnitude of the resonance peak as this is not a second order system. Each quadratic term in the denominator contributes -2 to the composite slope at high frequencies. The numerator term contributes a slope of $+1$ at frequencies above $1/\tau$. As a result, the composite asymptotic curve will have a slope of -3 at high frequencies.

The numerator term causes the magnitude curve to break upward at $\omega = 1/\tau$. If $1/\tau$ is less than ω_{r1}, the curve will break upward before the -2 slope of the first quadratic term takes effect. Figure shown below represents the magnitude plot for three different cases, including the case where $\tau = 1.25$ s, which corresponds to a corner frequency of $\omega = 0.8$ rad/s. As compared to the case having no numerator dynamics ($\tau = 0$), the choice of $\tau = 1.25$ s can be seen to raise the magnitude. Using a value of $1/\tau$ that is larger than the smallest resonance frequency will not increase the amplitude ratio because the -2 slope from the first quadratic term will take effect before the $+1$ slope of the numerator term makes its contribution. An example of this is shown in the Figure for $\tau = 0.0125$ s, which correspond to a corner frequency of 80 rad/s.



Problem 5

These are both second order systems. The first system ($G_1(s)$) has a complex conjugate pole pair with undamped natural frequency of $\omega_0 = 1$ rad/s and damping ratio $\zeta = 0.25$. The second system ($G_2(s)$) has a repeated real pole with $\tau = 1$ s. The first system displays resonance as $\zeta < 0.707$.

The Bode and Nyquist plots for both systems is shown below. The increase in magnitude is visible around the resonance frequency of the first system $\omega_r = 0.93$ rad/s while the magnitude of the second system monotonically decreases with increasing frequency. Red lines show the asymptotes, blue curves are for the first system, and green curves are for the second system. The slope of the red line is -2.

The phase plots are also different as the arctan function becomes steeper (sigmoidal) with decreasing damping ratio.

The changes in magnitude and phase plots impact the Nyquist plot. The curve will trace a different trajectory, getting outside the unit circle for the first system. The dashed circle in the Nyquist plot denotes the unit circle.